

# Pointed computations and Martin-Löf randomness\*

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*For Barry, for whom the magnificent incomputability of the world was a deeply held belief.  
The only response? An attempt at understanding this chaos at a higher order.*

**Abstract.** Schnorr showed that a real  $X$  is Martin-Löf random if and only if  $K(X \upharpoonright_n) \geq n - c$  for some constant  $c$  and all  $n$ , where  $K$  denotes the prefix-free complexity function. Fortnow (unpublished) and Nies, Stephan and Terwijn [NST05] observed that the condition  $K(X \upharpoonright_n) \geq n - c$  can be replaced with  $K(X \upharpoonright_{r_n}) \geq r_n - c$ , for any fixed increasing computable sequence  $(r_n)$ , in this characterization. The purpose of this note is to establish the following generalisation of this fact. We show that  $X$  is Martin-Löf random if and only if  $\exists c \forall n K(X \upharpoonright_{r_n}) \geq r_n - c$ , where  $(r_n)$  is any fixed *pointedly*  $X$ -computable sequence, in the sense that  $r_n$  is computable from  $X$  in a self-delimiting way, so that at most the first  $r_n$  bits of  $X$  are queried in the computation of  $r_n$ . On the other hand, we also show that there are reals  $X$  which are very far from being Martin-Löf random, but for which there exists some  $X$ -computable sequence  $(r_n)$  such that  $\forall n K(X \upharpoonright_{r_n}) \geq r_n$ .

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# 1 Introduction

A well known result of Schnorr (see Chaitin [Cha75]) is that Martin-Löf's notion of algorithmic randomness from [ML66] can be expressed in terms of incompressibility with respect to prefix-free machines. In particular, a real  $X$  is Martin-Löf random if and only if  $\exists c \forall n K(X \upharpoonright_n) > n - c$ , where  $K$  denotes the prefix-free complexity function. The latter condition says that there exists a constant  $c$  such that all the initial segments of  $X$  are  $c$ -incompressible (in a prefix-free sense). As reported in Downey and Hirschfeldt [DH10, Proposition 6.1.4], Fortnow (unpublished) and Nies, Stephan and Terwijn [NST05] showed that Schnorr's characterisation remains valid if we replace the condition  $\exists c \forall n K(X \upharpoonright_n) > n - c$  with  $\exists c \forall n K(X \upharpoonright_{r_n}) > r_n - c$ , where  $(r_n)$  is any computable increasing sequence.

In this note we consider the extent to which this fact can be generalised to incomputable increasing sequences  $(r_n)$ . It is well known that there are reals which are not Martin-Löf random, yet have infinitely many 0-incompressible initial segments. Hence this characterisation does not hold for arbitrary increasing sequences  $(r_n)$ . We consider the case when  $(r_n)$  is computable from  $X$ . Our main result is that if  $(r_n)$  is computable from  $X$  in a certain restricted way, then  $X$  is Martin-Löf random if and only if  $\exists c \forall n K(X \upharpoonright_{r_n}) > r_n - c$ . On the other hand we show how to construct reals  $X$  and  $X$ -computable sequences  $(r_n)$  such that the above equivalence fails, so we have  $\forall n K(X \upharpoonright_{r_n}) \geq r_n$  but  $X$  is very far from being Martin-Löf random.

## 1.1 Pointed computability

Our main result is that if  $X$  computes  $(r_n)$  in a certain natural fashion, then  $\exists c \forall n K(X \upharpoonright_{r_n}) > r_n - c$  is a sufficient and necessary condition for the Martin-Löf randomness of  $X$ . In this section we formalise the notion of oracle-computability required in order for this equivalence to hold, which we call pointed computability.

A Turing functional  $\Phi$  can be thought of as a machine which takes as inputs a number  $n$  and a program  $\sigma$ , and either halts on these inputs producing a number  $\Phi^\sigma(n)$  as output, or else diverges. The consistency of  $\Phi$  requires that if  $\rho_0 \subseteq \rho_1$  and  $\Phi^{\rho_0}(n) \downarrow$ , then the computation  $\Phi^{\rho_1}(n)$  is identical to that for  $\Phi^{\rho_0}(n)$ , yielding the same output. Then  $\Phi^X(n)$  can be defined as  $\lim_s \Phi^{X \upharpoonright_s}(n)$ . Without loss of generality, given a Turing functional  $\Phi$ , a string  $\rho$  and a number  $n$ , we may assume that  $\Phi^\rho(n) \downarrow$  implies  $|\rho| \geq n$  and  $|\rho| \geq \Phi^\rho(n)$ . This is a standard convention and it is not hard to see that if  $\Psi^X = Z$  for two reals and a Turing functional  $\Psi$  which might not obey the convention, then there exists a Turing functional  $\Phi$  which does obey the stated convention and for which  $\Phi^X = Z$ .

Given a Turing functional  $\Phi$ , a number  $n$  and strings  $\rho_0, \rho_1$ , the consistency property says that if  $\Phi^{\rho_0}(n) \neq \Phi^{\rho_1}(n)$  then we must have that  $\rho_0 \upharpoonright_{|\rho_1|} \neq \rho_1$ , i.e. the finite oracles differ at a digit which is less than  $\min\{|\rho_0|, |\rho_1|\}$ . The following definition is based on a more stringent consistency requirement.

**Definition 1.1** (Pointed computations). *Given a real  $X$ , we say that a sequence  $(r_n)$  is pointedly  $X$ -computable if  $(r_n)$  is (strictly) increasing and there exists a Turing functional  $\Phi$  such that  $\Phi^{X \upharpoonright_{r_n}}(n) \downarrow = r_n$  for each  $n$ .*

The latter condition in Definition 1.1 says that some oracle Turing machine computes each  $r_n$  from  $X$  with oracle-use bounded above by  $r_n$ . Note that, without loss of generality, we may assume that the oracle-use in this computation is exactly  $r_n$ .

As a typical example of pointed computations (suppressing the monotonicity requirement for now), consider an oracle machine which starts on input  $n$  by reading increasingly longer initial segments of the oracle tape, and eventually stops after  $s > n$  steps, with output  $s$ . If, for a given  $X$ , such a machine converges for every  $n$ , then it produces a pointed computation in the sense of Definition 1.1. Another example is the settling time of a non-computable c.e. set  $A$ : let  $r_0 = 0$  and for each  $n$  let  $r_{n+1}$  be the least number which is larger than  $r_n$  and such that  $A_s \upharpoonright_n = A \upharpoonright_n$  for all  $s \geq r_{n+1}$ . Then  $(r_i)$  is pointedly  $A$ -computable and non-computable.

Later we will note that there are weaker notions of computability which suffice for our characterization of Martin-Löf randomness. One such notion is the condition that  $K(r_n \mid \tau) = \mathbf{O}(1)$  for all  $n$  and all  $\tau \supseteq X \upharpoonright_{r_n}$ . Note that the latter is the non-uniform version of the notion of Definition 1.1.

## 1.2 Our results

Our main result is the following, which we prove in Section 2.

**Theorem 1.2** (Randomness condition). *Suppose that  $(r_n)$  is pointedly computable from a real  $X$ . Then  $X$  is Martin-Löf random if and only if  $\exists c \forall n K(X \upharpoonright_{r_n}) > r_n - c$ .*

It is well-known that there are reals which are not Martin-Löf random, yet have infinitely many incompressible initial segments. Hence Theorem 1.2 does not hold if we simply waive the requirement that  $(r_n)$  is pointedly computable from  $X$ . One may ask, however, if Theorem 1.2 continues to hold if we merely require that  $(r_n)$  is computable from  $X$  and not that it is pointedly  $X$ -computable. It is not surprising that the latter question has a negative answer. One way to exhibit an example witnessing this fact, is to construct a real with infinitely many incompressible initial segments, which computes the halting problem and is not Martin-Löf random. Since the prefix-free complexity function is computable from the halting problem  $\emptyset'$ , given such an oracle  $X$  we can effectively find infinitely many  $t$  such that  $K(X \upharpoonright_t) \geq t - c$ . This gives the following fact.

**Proposition 1.3.** *Suppose that  $X$  computes the halting problem,  $X$  is not Martin-Löf random and there exists some constant  $c$  and infinitely many  $n$  such that  $K(X \upharpoonright_n) \geq n - c$ . Then  $X$  computes an increasing sequence  $(r_n)$  such that  $K(X \upharpoonright_{r_n}) \geq r_n - c$  for all  $n$ .*

In Section 3 we present two ways of constructing oracles  $X$  which have the properties mentioned in Proposition 1.3, thus establishing the following.

**Theorem 1.4.** *There exists a real  $X$  and an  $X$ -computable increasing sequence  $(r_n)$ , such that  $r_n < K(X \upharpoonright_{r_n})$  for all  $n$ , and  $X$  is not Martin-Löf random.*

Our first construction of such  $X$  involves starting from a Martin-Löf random  $Y$  which computes the halting problem, and inserting zeros at certain places, thus causing  $X$  to be non-random, while preserving its ability to calculate lengths at which its initial segments have high prefix-free complexity. The second construction of such an oracle  $X$  is more flexible, and gives a real which is highly non-random, in the sense that its characteristic sequence has a computable subsequence of zeros.

### 1.3 Related concepts and results from the literature

A central notion studied in this paper is that of a set  $X$  which is able to compute a sequence of positions in its binary expansion where the corresponding initial segments are incompressible. Clearly every Martin-Löf random has this property, but there are also many reals with this property which are not Martin-Löf random. This notion might remind some readers of the *autocomplex reals* (see [KHMS06, KHMS11] or [DH10, Section 8.16]), which are the reals  $X$  which compute a non-decreasing unbounded function  $f$  such that  $K(X \upharpoonright_n) \geq f(n)$  for all  $n$ . Moreover, Theorem 1.2 has similarities to a result by Miller and Yu [MY10], which says that if  $\sum_i 2^{-g(i)} < \infty$  and  $g$  is computable from  $X$  with identity oracle-use, then  $\exists c \forall n K(X \upharpoonright_n) \leq n + g(n) + c$ . Note that the latter result can be seen as an extension of the following consequence of the Kraft-Chaitin inequality:

$$\text{if } g \text{ is computable and } \sum_i 2^{-g(i)} < \infty \text{ then there exists a constant } c \text{ such that for} \quad (1.3.1) \\ \text{all } X \text{ and all } n \text{ we have } K(X \upharpoonright_n) \leq n + g(n) + c.$$

It is clear that the result of Miller and Yu [MY10] is related to its special case (1.3.1) in the same way as our Theorem 1.2 is related to results originally obtained by Fortnow (unpublished) and Nies, Stephan and Terwijn [NST05], discussed earlier.

## 2 Proof of Theorem 1.2

We need to show that if  $(r_n)$  is pointedly  $X$ -computable, then  $X$  is Martin-Löf random if and only if  $\exists c \forall n K(X \upharpoonright_{r_n}) > r_n - c$ .

The ‘only if’ direction in this statement is trivial, so it remains to show the ‘if’ direction. We prove the contrapositive. Assuming that  $X$  is not Martin-Löf random and  $(r_n)$  is pointedly  $X$ -computable, we show that for each constant  $c$  there exists some  $n$  such that  $K(X \upharpoonright_{r_n}) \leq r_n - c$ . Let  $\Phi$  be a Turing functional such that  $\Phi^X(n) = r_n$  for all  $n$ , and such that for every  $n, r, \rho$  such

that  $\Phi^\rho(n) \downarrow = r$  we have  $\Phi^{\rho \upharpoonright r}(n) \downarrow$ ,  $|\rho| > n$ ,  $|\rho| \geq r$  and  $\Phi^\rho(i) \downarrow$  for all  $i < n$ . Let  $U$  be the underlying optimal universal prefix-free machine.

We define a prefix-free machine  $M$  which makes use of the descriptions of  $U$  as follows. For each vector  $(\rho, \sigma, n, t)$ , such that:

- $\Phi^\rho(i) \downarrow$  for all  $i \leq n+1$  and  $\Phi^\rho(i) < \Phi^\rho(i+1)$  for all  $i < n+1$ ;
- $t \in (\Phi^\rho(n), \Phi^\rho(n+1)]$  and  $U(\sigma) = \rho \upharpoonright t$ ;

let  $\tau$  be the digits of  $\rho$  between digit  $t$  and digit  $\Phi^\rho(n+1)$  and note that

$$U(\sigma) * \tau = \rho \upharpoonright_{\Phi^\rho(n+1)}, \Phi^{U(\sigma)*\tau}(n+1) = \Phi^\rho(n+1) \text{ and } \Phi^{U(\sigma)*\tau}(n) = \Phi^\rho(n).$$

Let  $M$  describe  $\rho \upharpoonright_{\Phi^\rho(n+1)}$  with the string  $\sigma * \tau$ , defining  $M(\sigma * \tau) = \rho \upharpoonright_{\Phi^\rho(n+1)}$ . This completes the definition of  $M$ .

It follows immediately from the definition that  $M$  is effectively calculable. Next we show that  $M$  does not allocate two different strings the same description. Given two identical  $M$ -descriptions  $\sigma_0 * \tau_0 = \sigma_1 * \tau_1$ , since  $U$  is a prefix-free machine we must have  $\sigma_0 = \sigma_1$  and  $\tau_0 = \tau_1$ . By the construction of  $M$ , the string described in both cases is then  $U(\sigma_0) * \tau_0$ . In a similar manner, we may show that  $M$  is a prefix-free machine. Suppose that  $\sigma_0 * \tau_0 \subseteq \sigma_1 * \tau_1$  are two descriptions issued by  $M$ , resulting from the vectors  $(\rho_0, \sigma_0, n_0, t_0)$ ,  $(\rho_1, \sigma_1, n_1, t_1)$  respectively. Since  $U$  is a prefix-free machine and  $\sigma_0, \sigma_1$  are  $U$ -descriptions, we have  $\sigma_0 = \sigma_1$ ,  $t_0 = t_1$  and  $\tau_0 \subseteq \tau_1$ . Let  $\sigma$  be  $\sigma_0$  and let  $t$  be  $t_0$ . Then  $\rho_0 \subseteq \rho_1$  and hence<sup>1</sup>

$$\Phi^{\rho_0}(n_0+1) = \Phi^{U(\sigma)*\tau_0}(n_0+1) = \Phi^{U(\sigma)*\tau_1}(n_0+1) = \Phi^{\rho_1}(n_0+1). \quad (2.0.1)$$

Thus  $(\Phi^{\rho_0}(n_0), \Phi^{\rho_0}(n_0+1)] = (\Phi^{\rho_1}(n_0), \Phi^{\rho_1}(n_0+1)]$ , and  $t$  belongs to both intervals. This means that  $n_0 = n_1$ , because otherwise  $t_1$  would have to belong to a different interval, not the one determined by  $n_0$  and  $\rho_1$ , since the values of  $\Phi$  are strictly monotone. This would be a contradiction by the choice of  $(\rho_1, \sigma_1, n_1, t_1)$  in the definition of  $M$  and the fact that  $t_0 = t_1$  which was established earlier. So let  $n = n_0 = n_1$ . Since  $U(\sigma) * \tau_0 = \rho_0 \upharpoonright_{\Phi^{\rho_0}(n+1)}$  and  $U(\sigma) * \tau_1 = \rho_1 \upharpoonright_{\Phi^{\rho_1}(n+1)}$ , by (2.0.1) the strings  $U(\sigma) * \tau_0$ ,  $U(\sigma) * \tau_1$  have the same length, so  $\tau_0 = \tau_1$ , which shows that the two descriptions  $\sigma_0 * \tau_0$ ,  $\sigma_1 * \tau_1$  are identical. This completes the proof that  $M$  is a prefix-free machine.

It remains to show that if  $X$  is not Martin-Löf random, then for each  $c$  there exists some  $n$  with  $K(X \upharpoonright_{r_n}) \leq r_n - c$ . Let  $d$  be a constant such that  $K(\eta) \leq K_M(\eta) + d$  for all strings  $\eta$ . Given any constant  $c$ , since  $X$  is not Martin-Löf random, there exists some  $t > 0$  such that  $K(X \upharpoonright_t) \leq t - c - d$ . Let  $n$  be such that  $t \in (r_n, r_{n+1}]$ . Then  $M$  will describe  $X \upharpoonright_{r_{n+1}}$  with  $\sigma * \tau$

<sup>1</sup>An intuitive description of the argument that follows for showing that  $\tau_0 = \tau_1$  is this: each of  $\rho_0, \rho_1$  determine a sequence  $\Phi^{\rho_i}(i), i \leq n_0+1$ , although  $\rho_1$  may be able to define  $\Phi^{\rho_1}(i)$  for  $i > n_0+1$ . However  $t \in (\Phi^{\rho_0}(n_0), \Phi^{\rho_0}(n_0+1)]$  so by the construction of  $M$  we have that both  $\tau_0, \tau_1$  equal the digits of  $\rho_0$  (or equivalently  $\rho_1$ ) between digit  $t$  and digit  $\Phi^{\rho_0}(n_0+1)$ .

where  $\sigma$  is the shortest description of  $X \upharpoonright_t$  and the length of  $\tau$  is  $r_{n+1} - t$ . The length of  $\sigma$  is  $K(X \upharpoonright_t)$ , which is at most  $t - c - d$ . We have:

$$K_M(X \upharpoonright_{r_{n+1}}) \leq (t - c - d) + (r_{n+1} - t) = r_{n+1} - c - d.$$

So  $K(X \upharpoonright_{r_{n+1}}) \leq r_{n+1} - c$ , as required.

### 3 Proof of Theorem 1.4

As discussed in the introduction, we present two different constructions of a real  $X$  which computes an increasing sequence  $(r_n)$  such that  $K(X \upharpoonright_{r_n}) > r_n$  for all  $n$  and  $X$  is not Martin-Löf random.

#### 3.1 An ad hoc construction

One way to construct a real  $X$  with the property of Proposition 1.3 is to start from a Martin-Löf random real  $Y$  which computes the halting problem, and insert zeros into  $Y$  in a way that does not change the fact that  $\emptyset'$  is computable from the resulting oracle, but does ensure non-randomness. Recall that a Martin-Löf random real  $Y$  which computes the halting problem exists by the Kučera-Gács theorem [Kuč85, Gács86]. In this construction we use the result of Chaitin [Cha75], which asserts that:

$$\text{if } Z \text{ is Martin-Löf random then } \lim_s (K(Z \upharpoonright_s) - s) = \infty. \quad (3.1.1)$$

We also use the fact that for each real  $Z$  which is Martin-Löf random and each string  $\sigma$ , the real  $\sigma * Z$  is Martin-Löf random, and the fact from [Cha87] that:

$$\begin{aligned} &\text{if } Z \text{ is Martin-Löf random and } f \text{ is a partial computable function on} \\ &\text{strings, then if } f(Z \upharpoonright_n) \downarrow \text{ for infinitely many } n, \text{ there are infinitely many} \\ &t \text{ such that } f(Z \upharpoonright_t) \downarrow \neq Z(t). \end{aligned} \quad (3.1.2)$$

The reader may observe that a partial computable prediction rule  $f$  as above which is always successful on  $Z$  would give rise to a computable martingale which succeeds on  $Z$ , which we know is not possible for Martin-Löf random reals (e.g. see [DH10, Section 6.3]).

Let  $\Phi$  be a functional via which  $Y$  computes the complexity function  $K$ , i.e. such that  $\Phi^Y(\sigma) = K(\sigma)$  for all  $\sigma$ . We form  $X$  from  $Y$  by inserting 0s at various positions, in a stage by stage process. The real  $X$  is defined as the limit of a sequence  $X_s$  and we form each  $X_{s+1}$  from  $X_s$  by inserting a 0 at position  $t_{s+1}$ , which means that we define  $X_{s+1}(n) = X_s(n)$  for  $n < t_{s+1}$ ,  $X_{s+1}(n) = 0$  for  $n = t_{s+1}$  and  $X_{s+1}(n+1) = X_s(n)$  for  $n \geq t_{s+1}$ . At stage 0 we define  $X_0 = Y$ , and (for convenience) define  $t_0 = -1$ . Now inductively suppose that we have performed stages  $0, \dots, k$ , and that we have recorded  $t_0, \dots, t_k$ . For any string  $\tau \subset X_k$ , let  $\tau^*$  be the string which

results from removing all of the 0s that we have inserted during the stages  $\leq k$ . At step  $k+1$  we search for  $\sigma \subset X_k$  of length  $> t_k + 2$  and  $\tau \subset X_k$  with  $\sigma \subset \tau$  such that  $\Phi^{\tau^*}$  computes  $K(\sigma)$  and  $K(\sigma) > |\sigma|$ . By (3.1.1), it follows that such  $\sigma$  and  $\tau$  exist. Then we define  $t_{k+1} = |\tau|$  and insert a 0 at position  $t_{k+1}$ .

This completes the construction of  $X$  given  $Y$ . From the construction it follows that  $X$  computes both  $Y$  and the sequence  $(t_k)$ . This follows because  $X$  is able to retrace the construction. Inductively suppose that  $X$  has been able to retrace the construction up until the end of stage  $k$ , and so knows the values  $t_0, \dots, t_k$ . Then, using the oracle for  $X$  we can perform the same search that was carried out at stage  $k+1$ , but using  $X$  rather than  $X_k$ : we search for  $\sigma \subset X$  of length  $> t_k + 2$  and  $\tau \subset X$  with  $\sigma \subset \tau$  such that  $\Phi^{\tau^*}$  computes  $K(\sigma)$  and  $K(\sigma) > |\sigma|$ . Since the next zero is inserted after  $\tau$ , the result of the search is the same as when  $X_k$  was used during the construction at stage  $k+1$ . Then  $t_{k+1} = |\tau|$ , completing the induction step. Therefore  $X$  computes the halting set  $\emptyset'$ . Moreover there is a partial computable function  $f$  such that for each  $\sigma \subset X$  we have  $f(\sigma) \downarrow$  if and only if  $\sigma = X \upharpoonright_{t_k}$  for some  $k$  ( $f$  uses  $\sigma$  as an oracle to try and retrace the construction and converges on  $\sigma$  if it is of length  $t_k$  for some  $k$  in the retraced construction). For each  $k$ , the next digit  $t_k$  of  $X$  is a 0, so  $f$  is a partial computable prediction rule that succeeds on  $X$ , which means that  $X$  is not Martin-Löf random. This completes the construction of a set  $X$  with the properties of Proposition 1.3.

### 3.2 A refined construction

Here we construct the required  $X$  by finite extensions. This construction can be combined with other requirements. For example,  $X$  can be highly non-random, in the sense that it has a computable sequence of 0s. We need some facts from the theory of prefix-free Kolmogorov complexity. For each string  $\sigma$ , let  $\sigma^*$  denote the shortest prefix-free description of  $\sigma$  (if there are many shortest descriptions, we consider the one which describes  $\sigma$  first). Also let  $K(\tau \mid \rho)$  denote the prefix-free complexity of  $\tau$  relative to string  $\rho$ . The following is a relativized version of Chaitin's counting theorem from [Cha75].

**Lemma 3.1** (Relativized counting theorem). *There exists a constant  $c$  such that for all  $\sigma, r$  and all  $n > \sigma$ :*

$$\left| \left\{ \tau \mid \sigma \subseteq \tau \wedge \tau \in 2^n \wedge K(\tau \mid \sigma^*) \leq |\tau| - r - K(\sigma) \right\} \right| \leq 2^{n-K(\sigma)+c-r-K(n \mid \sigma^*)}.$$

**Proof.** Given  $\sigma$  we define  $F(n \mid \sigma^*)$  for  $n > |\sigma|$  to be the  $-\log$  of the weight of the prefix-free descriptions relative to  $\sigma^*$  which describe extensions of  $\sigma$  of length  $n$ . Then since the relative prefix-free complexity  $K$  is a minimal information measure, there exists a constant  $c$  such that for all  $n, \sigma$ ,

$$2^{-F(n \mid \sigma^*)} < 2^{-K(n \mid \sigma^*)+c}. \quad (3.2.1)$$

We claim that the constant  $c$  has the property of the statement of the lemma. For a contradiction, suppose that this is not the case. Then for some  $n$  there are more than  $2^{n-K(\sigma)+c-r-K(n \mid \sigma^*)}$  many

$\tau$  such that  $K(\tau \mid \sigma^*) \leq |\tau| - r - K(\sigma)$ . In that case we have

$$2^{-F(n \mid \sigma)} > 2^{n-K(\sigma)+c-r-K(n \mid \sigma^*)} \cdot 2^{-n+r+K(\sigma)} = 2^{c-K(n \mid \sigma^*)},$$

which contradicts (3.2.1). This contradiction concludes the proof of the lemma.  $\square$

Recall the symmetry of information fact from [Gács74, Cha75]:

$$K(\tau) + K(\sigma \mid \tau^*) = K(\sigma) + K(\tau \mid \sigma^*) + \Omega(1)$$

where  $f = g + \Omega(1)$  for two functions  $f, g$  means that  $|f(n) - g(n)|$  is bounded above. Since  $K(\tau) = K(\sigma, \tau) + \mathbf{O}(1)$  for all strings  $\sigma, \tau$ , by the symmetry of information, Lemma 3.1 has the following corollary.

**Corollary 3.2** (Relativized counting, again). *There exists a constant  $c$  such that*

$$\left| \left\{ \tau \mid \sigma \subseteq \tau \wedge \tau \in 2^n \wedge K(\tau) \leq |\tau| - r \right\} \right| \leq 2^{n-K(\sigma)+c-r-K(\sigma \mid \tau^*)-K(n \mid \sigma^*)}$$

for all  $\sigma, r$  and all  $n > \sigma$ .

Corollary 3.2 is the tool we are going to use for our finite extension construction. The problem we face is, given a string  $\sigma$  to find an extension  $\tau$  such that  $K(\tau) > |\tau|$ . Since there are  $2^{n-|\sigma|}$  many extensions of  $\sigma$  of length  $n$ , by Corollary 3.2 it suffices to consider  $n$  such that  $2^{n-K(\sigma)+c-K(n \mid \sigma^*)} < 2^{n-|\sigma|}$ , which means that  $|\sigma| + c < K(\sigma) + K(n \mid \sigma^*)$  so

$$K(n \mid \sigma^*) > |\sigma| - K(\sigma) + c.$$

Such a number  $n$  clearly exists in the interval  $[|\sigma|, |\sigma| + 2^{c+|\sigma|-K(\sigma)}]$ . The quantity  $|\sigma| - K(\sigma)$  is sometimes called the *randomness deficiency* of  $\sigma$ . We have shown that:

$$\begin{aligned} &\text{There exists a constant } c \text{ such that each } \sigma \text{ can be extended by less than } 2^{c+|\sigma|} - 1 \\ &\text{many bits to a string } \tau \text{ with } K(\tau) > |\tau|. \end{aligned} \quad (3.2.2)$$

We are ready to construct the required real  $X$  by finite extensions

$$\rho_0 \subset \tau_1 \subset \rho_1 \subset \tau_2 \subset \dots \quad (3.2.3)$$

In this construction the lengths  $\ell_i := |\tau_i|$  will be computable while the strings  $\rho_i$  will be chosen so that  $K(\rho_i) > |\rho_i|$  for all  $i$ . For each  $i$  the string  $\tau_{i+1}$  will be the concatenation of  $\rho_i$  with a string  $10 \dots 0$  such that  $|\tau_i| = \ell_i$ . So for each  $i$  the string  $\rho_i$  will be uniformly computable from  $\tau_i$ : simply find the first 1 starting from position  $|\tau_i|$  in  $\tau_i$  and moving to the left, and if this 1 is at position  $t$  then  $\rho_i = \tau_i \upharpoonright_t$ . Let  $c$  be the constant in (3.2.2).

Let  $\rho_0$  be the empty string  $\lambda$ , so that  $K(\rho_0) > |\rho_0|$ . Let  $\tau_1$  be the string  $\rho_0 * 10$  and let  $\ell_1 = |\rho_0| + 2 = 2$ . Note that the last digit of  $\tau_1$  is a 0. By (3.2.2) there exists an extension  $\rho_1$  of  $\tau_1$  such that  $|\rho_1| - |\tau_1| < 2^{c+\ell_1} - 1$  and  $K(\rho_1) > |\rho_1|$ . Then let  $\tau_2$  be the concatenation of  $\rho_1$



with a string  $10 \dots 0$  such that the length  $\ell_2 := |\tau_2|$  is  $\ell_1 + 2^{c+\ell_1}$ . Note that  $\tau_2$  is longer than  $\rho_1$  by at least 2 bits, so the last digit of  $\tau_2$  is a 0. Similarly, we can choose an extension  $\rho_2$  of  $\tau_2$  of less than  $2^{c+\ell_2} - 1$  many additional bits such that  $K(\rho_2) > |\rho_2|$ . As before we let  $\tau_3$  be the concatenation of  $\rho_2$  with a string  $10 \dots 0$  such that the length of  $\tau_3$  is  $\ell_3 := \ell_2 + 2^{c+\ell_2}$ . Note that  $\tau_3$  is longer than  $\rho_2$  by at least 2 bits, so the last digit of  $\tau_3$  is a 0.

The construction continues similarly, thus defining the computable sequence of lengths  $\ell_{n+1} = \ell_n + 2^{c+\ell_n}$  for each  $n > 0$ , where  $\ell_1 = 2$ , and the sequences (3.2.3) such that for all  $i > 0$  we have  $K(\rho_i) > |\rho_i|$ ,  $|\tau_i| = \ell_i$  and the last digit of  $\tau_i$  is 0. If we let  $X$  be the infinite extensions of the strings (3.2.3) then we have  $X(\ell_i - 1) = 0$  for all  $i$ , so  $X$  is not Martin-Löf random. On the other hand  $X \upharpoonright_{\ell_i} = \tau_i$  for all  $i$ , and since  $\tau_i$  uniformly computes  $\rho_i$ , we have that the sequence  $(|\rho_i|)$  is  $X$ -computable. Finally  $K(X \upharpoonright_{|\rho_i|}) = K(\rho_i) > |\rho_i|$  for all  $i$ , which concludes the verification of the required properties of  $X$ .

## 4 Conclusion and discussion

We have generalized the criterion for Martin-Löf randomness by Fortnow (unpublished) and Nies, Stephan and Terwijn [NST05], which says that if  $X$  has incompressible segments of a computable sequence of lengths  $(r_n)$ , then it is Martin-Löf random. We proved that the condition that  $(r_n)$  is computable can be replaced by the weaker condition that  $(r_n)$  is pointedly  $X$ -computable, in the sense that  $r_n$  is uniformly computable from any extension of  $X \upharpoonright_{r_n}$ . It is a simple exercise to extend our proof of this fact in order to replace the condition of pointed computation with a non-uniform version of it, namely that  $K(r_n \mid \tau) = \mathbf{O}(1)$  for all  $n$  and all  $\tau \supseteq X \upharpoonright_{r_n}$ . On the other hand, we showed that this condition is no longer sufficient for Martin-Löf randomness, if we merely require that  $r_n$  be computable from  $X$ . It would be interesting to refine this analysis and find exactly what kind of computations are allowed of a sequence  $(r_n)$  from an oracle  $X$  such that the Martin-Löf randomness of  $X$  is equivalent to the segments  $X \upharpoonright_{r_n}$  being incompressible.

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